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Hydrodynamic Coefficients of an Oscillating Ellipsoid Moving in the Free Surface

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The frequency dependent heave, pitch, sway, and yaw hydrodynamic coefficients associated with an oscillating ellipsoid traveling with forward speed in the free surface are evaluated from a three-dimensional potential flow analysis. The free-surface boundary condition in the mathematical model either includes the influence of forward speed or is simplified to the equivalent zero speed case. This variation produces a velocity potential which is either frequency and speed dependent or just frequency dependent. The influence of forward speed on all the hydrodynamic coefficients is discussed.

Nomenclature

A_{jk}		= hydrodynamic coefficient
A'_{jk}	$= A_{jk}/\rho(l/2)^3$	= nondimensional added mass $(j,k=1,2,3)$
	$= A_{jk}/\rho(l/2)^5$	= nondimensional added inertia $(j,k=4,5,6)$
	$= A_{jk}/\rho(l/2)^4$	= nondimensional cross-coupling coefficients $(j=1,2,3$ and $k=4,5,6$ or $j=4,5,6$ and $k=1,2,3)$
a		= wave amplitude
B_{jk}		= hydrodynamic coefficient
B'_{jk}	$= B_{jk}/\rho(l/2)^3(g/l)^{1/2}$	= nondimensional damping $(j,k=1,2,3)$
	$= B_{jk}/\rho(l/2)^5(g/l)^{1/2}$	= nondimensional damping $(j,k=4,5,6)$
	$= B_{jk}/\rho(l/2)^4(g/l)^{1/2}$	= nondimensional damping $(j=1,2,3$ and $k=4,5,6$ or $j=4,5,6$, and $k=1,2,3)$
F_n	$=\bar{U}/\sqrt{gl}$	= Froude number
G_j		= force or moment due to body motion
l		= length of body
$\underset{\sim}{n}$	$=n_1,n_2,n_3$	= outward unit normal
$\bar{S}$		= mean wetted surface area
t		= time
$\bar{U}$		= steady forward speed
β	$=\bar{U}\omega_e/g$	= nondimensional parameter
δ	$=\omega_e\sqrt{l/g}=\beta/F_n$	= nondimensional frequency parameter
η_j		$=(j=1,2,\ldots,6)$ displacements as defined in Fig. 1
ρ		= density of fluid
ϕ		= unsteady velocity potential
ϕ_k		= velocity potential associated with kth mode of motion
ϕ_0		= incident wave potential
ϕ_D		= diffracted wave potential
ω		= wave frequency
ω_e		= wave encounter frequency

Introduction

OVER the last three decades many linearized theoretical mathematical models have been proposed to describe the motions of a rigid ship in waves.[1-7] Generally, these formulations adequately describe the responses when the ship is at rest but are less successful when the ship moves at a prescribed forward speed. In the mathematical models, the influence of forward speed has been accounted for in a variety of ways, leading at times to quite different formulations and results. If slender body assumptions are adopted as the basis of development to a strip theory, the treatment of the influence of forward speed must be somewhat restricted. The possible forms of the mathematical model are discussed by Newman.[8,9] The advent of a fully three-dimensional analysis[10-13] has removed some of these restrictions and widened the scope for correctly including forward speed because some of the initial simplifying assumptions concerning forward speed are now redundant.

In this paper, a three-dimensional source distribution method is used[11-13] which includes two descriptions of the velocity potential. The first satisfies a simple free-surface condition which is equivalent to the condition associated with zero forward speed and hence is independent of forward speed. This produces a velocity potential which is essentially dependent on only the frequency of oscillation. The second formulation satisfies a speed-dependent free-surface boundary condition and this leads to a velocity potential which is both speed and frequency dependent. To illustrate the difference between these two formulations, calculations are presented of the hydrodynamic coefficients associated with heave, pitch, sway, and yaw motions of a half-immersed ellipsoid which has a length to beam ratio of 8 and travels with forward speeds defined by the Froude numbers $F_n=0,0.25$, and 0.5. The influence of forward speed is clearly illustrated and the Timman-Newman relationships[14] are again shown to be satisfied within the accuracy of the numerical method.

The General Problem

The mathematical formulation of the problem of a rigid body translating with constant mean forward velocity, $\bar{U}$ and oscillating with frequency of encounter ω_e in the free water surface is fully described by Newman[9,15] and only a brief description of the basic equations is included here.

In a linear theory, the unsteady velocity potential describing the translating, oscillating body may be expressed as[16]

$$\phi(x,y,z,t)=\left[\phi_0+\phi_D+\sum_{k=1}^{6}\eta_k\phi_k\right]e^{i\omega_e t}$$

where ϕ_0 is the incident wave potential, ϕ_D is the diffracted potential, and the components ϕ_k $(k=1,2,\ldots,6)$ are the radiation potentials due to motions of unit amplitude in each of the six degrees-of-freedom of motion having amplitudes η_1 (surge), η_2 (sway), η_3 (heave), η_4 (roll), η_5 (pitch), and η_6 (yaw) in the directions illustrated in Fig. 1.

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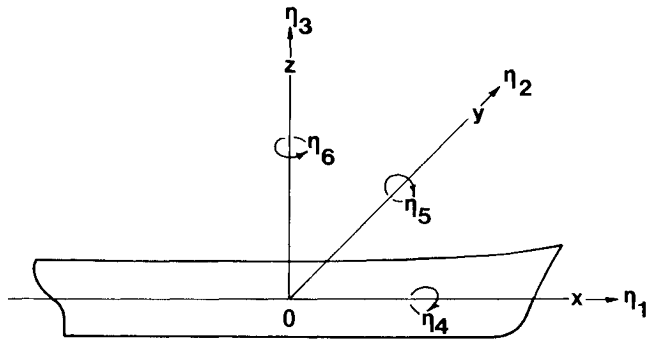

Fig. 1 Convention adopted for translations and angular rotations of ship about axis origin 0.

These potentials satisfy the conditions

$$\frac{\partial}{\partial n}(\phi_0+\phi_D)=0$$

on the mean submerged surface $\bar{S}$ and the boundary condition for the unsteady potential ϕ_j can be expressed as[5]

$$\frac{\partial\phi_j}{\partial n}=i\omega_e n_j+\bar{U}m_j \quad \text{on } \bar{S} \tag{1}$$

where the components

$$\left.\begin{array}{l} \underset{\sim}{n}=(n_1,n_2,n_3) \quad \underset{\sim}{r}=(x,y,z) \quad \underset{\sim}{r}\times\underset{\sim}{n}=(n_4,n_5,n_6) \\ m_1=0=m_2=m_3=m_4 \quad m_5=n_3 \quad m_6=-n_2 \end{array}\right\} \tag{2}$$

In the general formulation[9] the steady motion velocity potential is an integral part of the boundary conditions satisfied by the unsteady body potential and needs to be solved first. However, if it is assumed that the influence of the perturbation of the steady velocity potential in the analysis is neglected, then the simplified results of Eq. (2) are valid.

Under these conditions, the linearized free-surface boundary conditions satisfied by the unsteady potential is given by

$$\phi_{tt}-2\bar{U}\phi_{xt}+\bar{U}^2\phi_{xx}+g\phi_z=0 \quad \text{on } z=0 \tag{3}$$

where, for example, $\phi_{tt}=\partial^2\phi/\partial t^2$, etc.

The most general linear expressions for the hydrodynamic fluid actions caused by the body actions only is given by

$$G_j=-\rho\int\int_{\bar{S}} n_j(i\omega_e-\underset{\sim}{U}\cdot\nabla)\sum_{k=1}^{6}\eta_k\phi_k\,\mathrm{d}S=\sum_{k=1}^{6}T_{jk}\eta_k$$

for $j=1,2,\ldots,6$ and $\underset{\sim}{U}=(\bar{U},0,0)$. Here

$$T_{jk}=\omega_e^2A_{jk}-i\omega_eB_{jk}=-\rho\int\int_{\bar{S}} n_j(i\omega_e-\underset{\sim}{U}\cdot\nabla)\phi_k\,\mathrm{d}S \tag{4}$$

denotes the hydrodynamic action in the jth direction per unit oscillatory disturbance in the kth mode. The hydrodynamic coefficients A_{jk} and B_{jk} are associated with such disturbances.

Since the theory has placed no restrictions on the manner of solution of the velocity potentials, a three-dimensional source distribution method[11-13] is adopted. Initially, the hydrodynamic coefficients were calculated using Eq. (4) and involved the integration of $\partial\phi_k/\partial x$ over the hull surface. This procedure, however, may be subject to numerical inaccuracies due to difficulties in evaluating the velocity component in the longitudinal direction induced by a source at its own center.[11] A large number of elements are also required to accurately represent the fluctuations in $\partial\phi_k/\partial x$ over the wetted mean hull surface. However by applying Stokes's theorem[5-9] and assuming the ship slender, Eq. (4) reduces to

$$T_{jk}=-\rho\int\int_{\bar{S}}(i\omega_e n_j-\bar{U}m_j)\phi_k\,\mathrm{d}S \tag{5}$$

which is more amenable to evaluation.

Chang[10] used a similar mathematical formulation for the calculation of the hydrodynamic coefficients of a Series 60 ship model, although the numerical analysis adopted differs from the procedures employed in this paper.[11-13]

Velocity Potential

Simplified Form

The form of the velocity potential must be chosen to satisfy the conditions described previously. However, to simplify the analysis, the free-surface boundary condition described in Eq. (3) is suitably modified by assuming the frequency of oscillation ω_e to be large. This may be justified in the case of heave and pitch motions in head seas since it is the higher frequency region which is of interest. The terms in Eq. (3) involving the forward speed $\bar{U}$ are an order smaller than the remaining leading order terms and are negligible in comparison. The simplified free-surface condition reduces to

$$\phi_{tt}+g\phi_z=0 \quad \text{on } z=0 \tag{6}$$

corresponding to the zero forward speed condition in Eq. (3).

Since Eq. (6) is now independent of speed, the velocity potential may be expressed in terms of zero speed potentials. Using Eqs. (1,2, and 6) the speed dependent potential is defined as[6,7,11]

$$\phi_j=\phi_j^0 \qquad \text{for } j=1,2,3,4 \text{ (surge, sway, heave, roll)}$$

$$\begin{Bmatrix}\phi_5\\ \phi_6\end{Bmatrix}=\begin{Bmatrix}\phi_5^0\\ \phi_6^0\end{Bmatrix}\pm\frac{\bar{U}}{i\omega_e}\begin{Bmatrix}\phi_3^0\\ \phi_2^0\end{Bmatrix}\quad \begin{matrix}\text{for pitch}\\ \text{for yaw}\end{matrix} \tag{7}$$

where $\phi_j^0(j=1,2,\ldots,6)$ is a speed-independent potential satisfying the boundary conditions on the mean hull surface.[9]

More General Form

Although far more involved, a solution for the velocity potential may be obtained[10,13] based upon the general form of the free-surface boundary condition defined in Eq. (3) without resorting to the previously described simplification. A numerical procedure[13] has been developed for the evaluation of the form of the speed- and frequency-dependent source velocity potential which satisfies Eq. (3). The nature of this potential is complex and changes in character when the nondimensional parameter $\beta=\omega_e\bar{U}/g=\delta F_n=0.25$, where $\delta=(\omega_e\sqrt{l/g})$ denotes a nondimensional frequency parameter and F_n ($=\bar{U}/\sqrt{gl}$) is the Froude number. At this value of β, some of the terms in the potential are infinitely valued and a solution is not available. At zero speed [$\beta=0$ and the boundary condition reduces to Eq. (6)] waves radiate from the body in all directions, while at speed and $0<\beta<0.25$ there are four wave trains generated. That is three downstream and one upstream, for which the group velocity is greater than the speed of advance. For $\beta>0.25$, the upstream wave train disappears and the waves become concentrated in a wedge behind the source which narrows as β increases. For $\beta>1.63$, the waves due to the unsteady motion of the body are contained in the same wedge as those of the Kelvin wave pattern associated with the steady body motion.

Both types of velocity potential are defined in the Appendix and used to derive the results discussed in the following section. An examination of the numerical procedure adopted is omitted since it has been described in detail elsewhere.[11-13]

Calculations for an Ellipsoid

Calculations are presented of the hydrodynamic coefficients associated with the heave, pitch, sway, and yaw motions of a half-immersed ellipsoid moving on the free surface. The ellipsoid has a length to beam ratio of 8 and travels with forward speeds defined by Froude numbers $F_n = 0$, 0.25, and 0.5. The surface of the ellipsoid was represented by 52 panel elements as illustrated in Fig. 2 and a source distribution method[12,13] employed to derive the velocity potentials satisfying both of the free-surface boundary conditions discussed previously. Coefficients associated with these four motions are illustrated in Figs. 3-6, and the heave-pitch and sway-yaw cross-coupling coefficients are shown in Figs. 7 and 8, respectively. Where possible, the zero forward speed results have been compared with the exact analytic solutions derived by Kim.[17]

When the forward speed is zero, Eqs. (3) and (6) are identical and, as expected, the results predicted by both formulations coincide. In addition the results for heave and

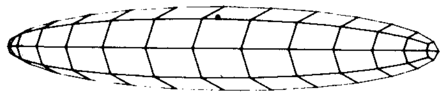

Fig. 2 Distribution of panel elements on surface of ellipsoid with length/beam ratio of 8.

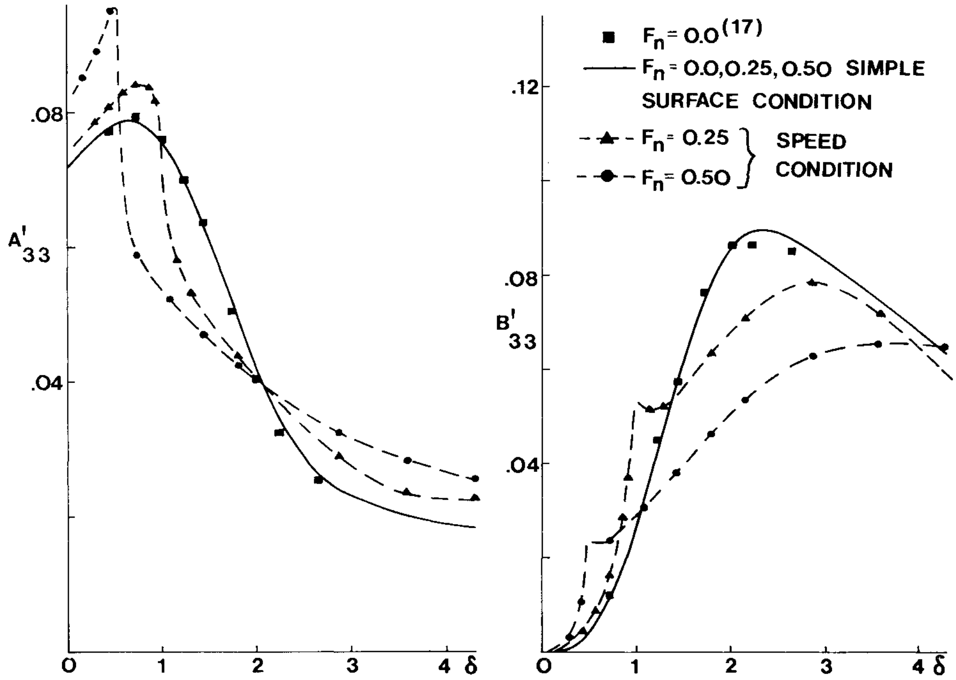

Fig. 3 Nondimensional coefficients associated with heave motion.

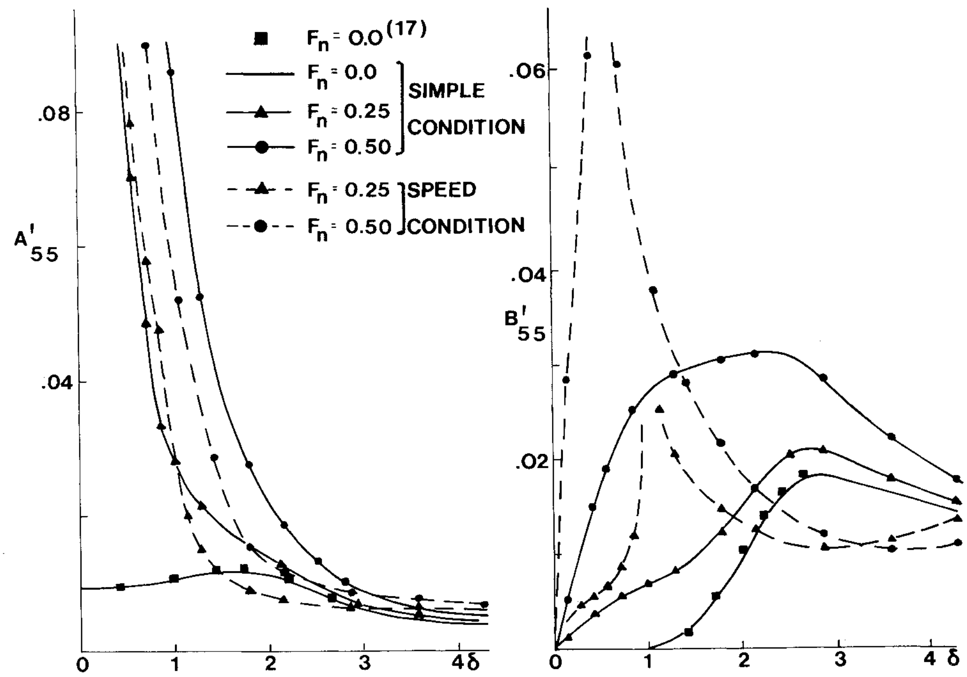

Fig. 4 Nondimensional coefficients associated with pitch motion.

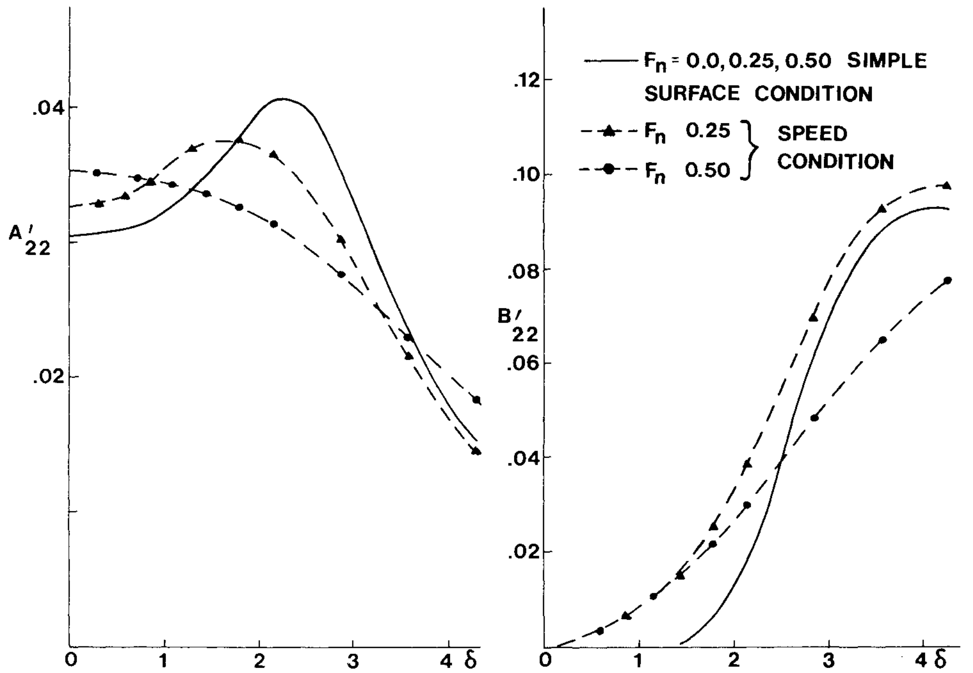

Fig. 5 Nondimensional coefficients associated with sway motion.

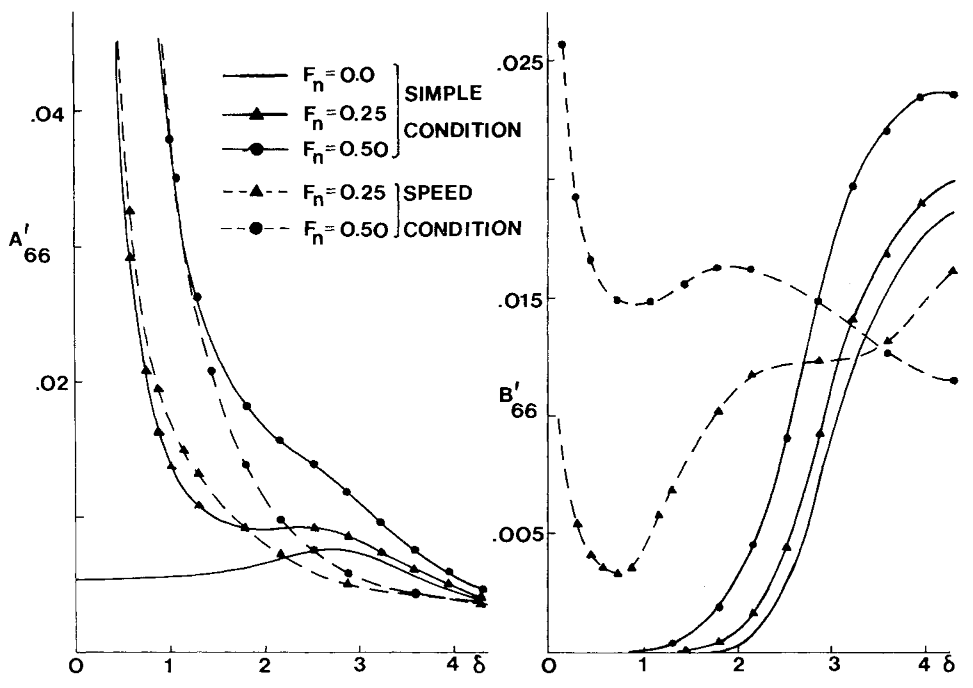

Fig. 6 Nondimensional coefficients associated with yaw motion.

pitch compare well with those of Kim[17] as shown in Figs. 3 and 4, and confirm the adequacy of the simple mesh chosen to represent the ellipsoid. It has been shown[12] that while a coarse mesh produces good results for heave and pitch coefficients, a much finer mesh is required to obtain satisfactory surge predictions.

For heave and sway motions, the simple free-surface condition analysis for the moving ellipsoid produces speed-independent added mass and damping coefficients as shown in Figs. 3 and 5, respectively. Equivalent results were obtained by using either Eq. (4) or (5) for the hydrodynamic actions. Since $m_2 = 0 = m_3$ in Eq. (5), this speed independence is partially expected whereas it is not obvious from an examination of Eq. (4). This indicates that the integration techniques employed to determine terms of the form $\partial\phi_k/\partial x$ for this slender body are adequate in sway and heave. In contrast, the hydrodynamic coefficients for the other two motions and all the cross-coupling terms show a marked dependence on speed as shown in Figs. 4 and 6-8.

With the inclusion of speed in the free-surface condition, the results presented in Figs. 3-8 indicate clearly the influence of this parameter. This is particularly noticeable for pitch and yaw motions where the nature of the dependence of the damping coefficients on frequency is significantly changed.

The heave and pitch damping coefficients exhibit a singularity at an oscillatory frequency corresponding to the critical value of $\beta = 0.25$ [i.e., $\delta = 1.0$ ($\omega_e = 0.7$) for $F_n = 0.25$ and $\delta = 0.5$ ($\omega_e = 0.35$) for $F_n = 0.50$]. A similar behavior has been indicated previously[18-20] and the corresponding added mass and inertia coefficients show a slight discontinuity through this critical value. The presence of a very localized singularity was impossible to investigate since the numerical

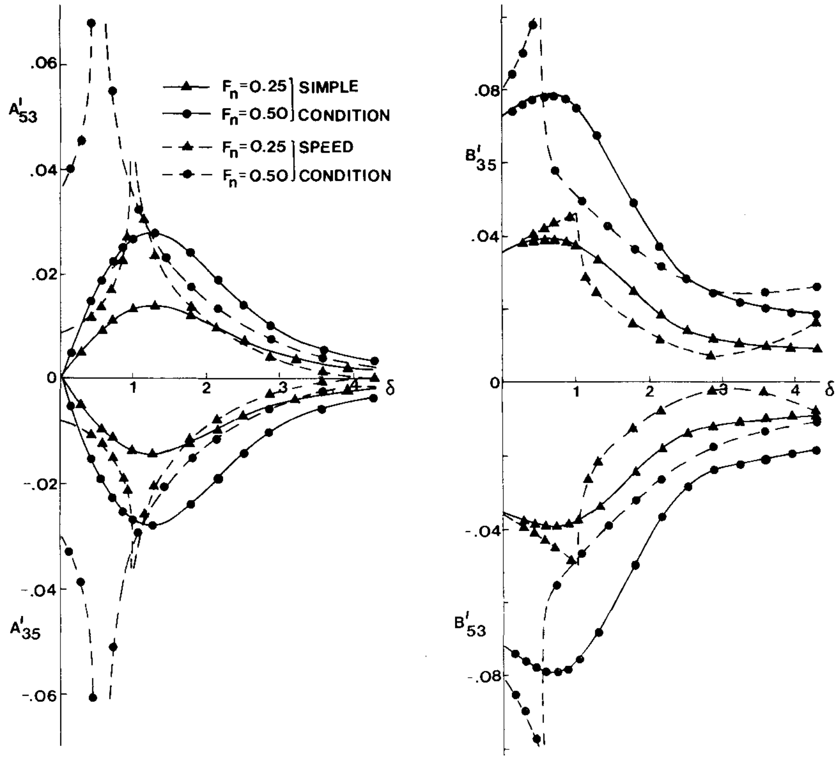

Fig. 7 Heave-pitch nondimensional cross-coupling coefficients.

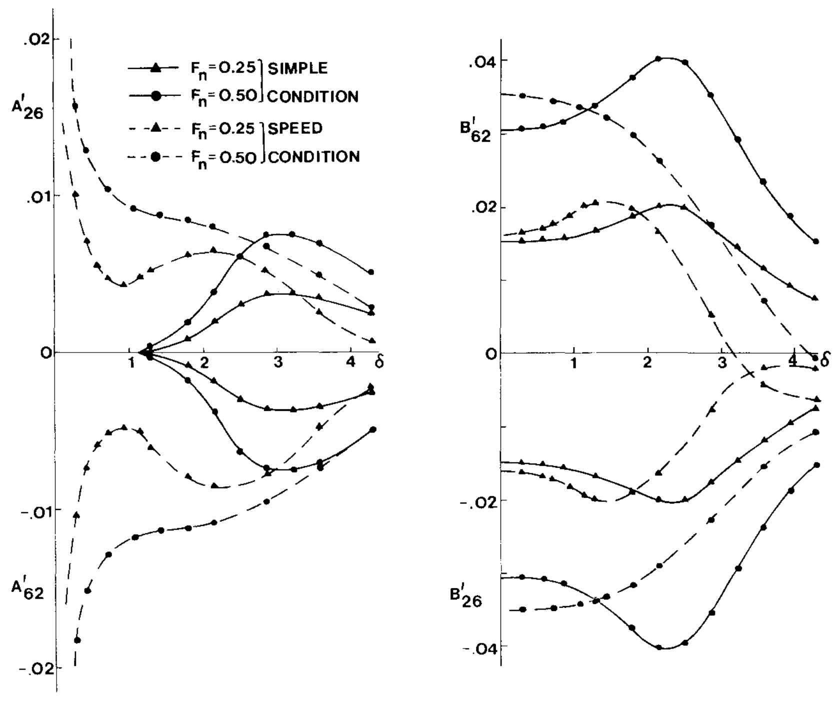

Fig. 8 Sway-yaw nondimensional cross-coupling coefficients.

procedure failed in the neighborhood of $\beta = 0.25$. The coefficients associated with the horizontal motions show no such discontinuous behavior near this critical value, as can be seen from Figs. 5 and 6. A similar conclusion was derived by Newman[20] for the damping of an oscillating ellipsoid situated near the free surface.

The results presented in Figs. 7 and 8 for the cross-coupling hydrodynamic coefficients associated with the moving ellipsoid determined by a simple free-surface condition analysis satisfy exactly the Timman-Newman[14] symmetry relationships. The heave-pitch coupling coefficients evaluated with the speed-dependent boundary condition analysis show discontinuous or singular behavior at the critical frequency corresponding to $\beta = 0.25$. These computed values do not satisfy the symmetry relationships exactly, but the qualitative trends in both sets of coefficients are similar. These differences may be attributable to the coarse mesh used to represent the surface of the ellipsoid since, to achieve symmetry in the cross-coupling terms, a finer mesh is required than that needed to predict the principal coefficients especially with increasing frequency. In the simple analysis, the velocity potential is solved for the zero speed body condition only and the speed dependence accounted for in the analysis as given by Eqs. (5) and (7). This insures that regardless of mesh size or accuracy, the necessary symmetry in the cross-coupling terms is always achieved.

Conclusions

The inclusion of the more complicated speed-dependent free-surface boundary condition in the analysis manifests itself in the forms of the calculated hydrodynamic coefficients. For example, the heave-added mass values A'_{33} vary with speed as has been observed from ship model oscillatory experiments,[21] whereas the calculation based upon the simpler free-surface condition is independent of speed.

The Timman-Newman relationships for the cross-coupling coefficients are exactly satisfied by the numerical analysis using the simple free-surface condition because of the mathematical form of the equations describing these coefficients. Numerical errors occur in the more complicated analysis and these relationships are not satisfied so perfectly. The number of elements describing the body and the frequency of oscillation are now important factors and govern the preciseness of the numerical solution. The qualitative description of the results obtained even by the coarse mesh adopted indicate that these relationships again remain valid—as they should be.

Although the more complex analysis requires far more computer time to operate than for the simple cases, it appears that such an analysis could be of use in predicting the motions of a ship operating in the high Froude number range.

Appendix: The Velocity Potentials

The velocity potential satisfying the *simple* free-surface boundary condition of Eq. (6) for a source at position (x_0,y_0,z_0) is given as[22]

$$\phi(x,y,z) = \frac{1}{R} + PV\int_0^\infty \frac{(k+\mu)}{\mu-k}\exp\{\mu(z+z_0)\}J_0(\mu r)\,\mathrm{d}\mu$$

$$-2\pi ik\exp\{k(z+z_0)\}J_0(kr)$$

where PV indicates that the integral is to be interpreted in the Cauchy principal value sense, $R=\{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2\}^{1/2}$, $r=\{(x-x_0)^2+(y-y_0)^2\}^{1/2}$, and for deep water the wave number $k=\omega^2/g$.

For the *speed-dependent* free-surface boundary condition defined in Eq. (3), the form of the velocity potential is[22]

$$\phi(x,y,z,) = \frac{1}{R_1} - \frac{1}{R_2} + \frac{2g}{\pi}\left\{\int_0^\gamma\int_0^\infty + \int_\gamma^{\pi/2}\int_{L_1}\right.$$

$$\left. + \int_{\pi/2}^{\pi}\int_{L_2}\right\} f(\theta,\mu)\,\mathrm{d}\mu\mathrm{d}\theta$$

where

$$f(\theta,\mu) = h(\theta,\mu)/[g\mu - (\omega_e + \bar{U}\mu\cos\theta)^2]$$

$$h(\theta,\mu)$$

$$=\mu\exp[\mu\{(z+z_0)+i(x-x_0)\cos\theta\}]\cos\{\mu(y-y_0)\sin\theta\}$$

$$\beta = \begin{cases} 0 & \text{if} \quad \beta=\omega_e\bar{U}/g<0.25 \\ \cos^{-1}(1/4\beta) & \text{if} \quad \beta\geq 0.25 \end{cases}$$

$$R_j^2 = (x-x_0)^2 + (y-y_0)^2 + \{z+(-1)^j z_0\}^2 \text{ for } j=1,2$$

The contour L_1 is from $\mu=0$ to $\mu=\infty$, passing under pole μ_1 and over pole μ_2. Contour L_2 is from $\mu=0$ to $\mu=\infty$, passing under both poles μ_3 and μ_4. The poles are positioned at

$$\left.\begin{array}{l}(g\mu_1)^{1/2} \\ (g\mu_3)^{1/2}\end{array}\right\} = \omega\left[\frac{1-(1-4\beta\cos\theta)^{1/2}}{2\beta\cos\theta}\right]$$

$$\left.\begin{array}{l}(g\mu_2)^{1/2} \\ -(g\mu_4)^{1/2}\end{array}\right\} = \omega\left[\frac{1+(1-4\beta\cos\theta)^{1/2}}{2\beta\cos\theta}\right]$$

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